Klausmeier, Christopher A., and Elena Litchman. 2001. Algal games: The vertical distribution of phytoplankton in poorly mixed water columns. Limnol. Oceanogr. 46: 1998–2007.

Web Appendix 1. Solving for Equilibrium

At equilibrium (denoted by ˆ), we set the time derivatives of Eq. 1 to zero.

$$0 = \hat{b}(z)\left(\min\left(f_I(\hat{I}(z)), f_R(\hat{R}(z))\right) - m\right) \tag{A1a}$$

$$0 = -\frac{\hat{b}(z)}{Y}\min\left(f_I(\hat{I}(z)), f_R(\hat{R}(z))\right) + D_R\frac{\partial^2 \hat{R}}{\partial z^2} + \varepsilon m\frac{\hat{b}(z)}{Y} \tag{A1b}$$

$$\left.\frac{\partial \hat{R}}{\partial z}\right|_{z=0} = 0 \tag{A1c}$$

$$\left.\frac{\partial \hat{R}}{\partial z}\right|_{z=z_b} = h\left(R_{\text{in}} - \hat{R}(z_b)\right) \tag{A1d}$$

$$\hat{I}(z) = I_{\text{in}}e^{-\int_0^z \left(a\hat{b}(Z)+a_{\text{bg}}\right)\, dZ} \tag{A1e}$$

We now solve for the equilibrium distributions of nutrients and light, and the equilbrium biomass.

To solve for the equilibrium distribution of nutrients in the water column, we note that under our assumption of a delta distribution of biomass, Eq. A1b reduces to

$$\frac{\partial^2 \hat{R}}{\partial z^2} = 0, z \neq z_l \tag{A2}$$

Equation A2 with boundary condition Eq. A1c shows that the diffusion of the nutrients equalizes the nutrient concentration above the layer,

$$\hat{R}(z) = \hat{R}(z_l),\ 0 \leq z \leq z_l \tag{A3}$$

Equation A2 also implies that nutrients vary linearly below the layer (denoted by z_l^+), thus

$$\left.\frac{\partial \hat{R}}{\partial z}\right|_{z=z_l^+} = \left.\frac{\partial \hat{R}}{\partial z}\right|_{z=z_b} = \frac{\hat{R}(z_b) - \hat{R}(z_l)}{z_b - z_l} \tag{A4}$$

Using Eq. A1d to solve Eq. A4 for $\hat{R}(z_b)$ we find

$$\hat{R}(z_b) = \frac{R_{\text{in}} h(z_b - z_l) + \hat{R}(z_l)}{1 + h(z_b - z_l)} \tag{A5}$$

so

$$\left.\frac{\partial \hat{R}}{\partial z}\right|_{z=z_l^+} = \left.\frac{\partial \hat{R}}{\partial z}\right|_{z=z_b} = \frac{R_{\text{in}} - \hat{R}(z_l)}{z_b + 1/h - z_l} \tag{A6}$$

Thus the distribution of nutrients at equilibrium is

$$\hat{R}(z) = \begin{cases} \hat{R}(z_l), & 0 \leq z \leq z_l \\ \hat{R}(z_l) + (z - z_l)\frac{R_{\text{in}} - \hat{R}(z_l)}{z_b + 1/h - z_l}, & z_l < z \leq z_b \end{cases} \tag{A7}$$

Nutrients are constant above the layer and increase linearly with depth below the layer (Fig. 1F).

Next, we will determine the equilibrium distribution of light. With our assumption of a delta distribution of biomass, Eq. A1e becomes

$$\hat{I}(z) = \begin{cases} I_{\text{in}} e^{-a_{\text{bg}} z}, & 0 \leq z < z_l \\ I_{\text{in}} e^{-a\hat{B} - a_{\text{bg}} z}, & z \geq z_l \end{cases} \tag{A8}$$

Light declines with depth exponentially due to background attenuation and drops a finite amount at the layer due to attenuation by phytoplankton (Fig. 1F). Note from Eq. A8 that a thin layer shades itself.

Now we will determine the equilibrium biomass. Equation A1a shows that phytoplankton reduce either the nutrient concentration or the light level to their break-even value at the layer since

$$\min\left(f_I(\hat{I}(z_l)), f_R(\hat{R}(z_l))\right) = m \tag{A9}$$

There are three possible cases: 1) the layer is nutrient-limited so that $\hat{R}(z_l) = R^*$ and $\hat{I}(z_l) > I^*$, 2) the layer is light-limited so that $\hat{I}(z_l) = I^*$ and $\hat{R}(z_l) > R^*$, or 3) the layer is limited by both resources so that $\hat{R}(z_l) = R^*$ and $\hat{I}(z_l) = I^*$.

In the case of nutrient-limitation, substituting Eq. A9 into Eq. A1b and rearranging,

$$\hat{b}(z) = \frac{Y D_R}{m(1-\varepsilon)} \frac{\partial^2 \hat{R}}{\partial z^2} \tag{A10}$$

integrating over z from 0 to z_b,

$$\hat{B} = \frac{YD_R}{m(1-\varepsilon)} \frac{\partial \hat{R}}{\partial z}\bigg|_{z=z_b} \tag{A11}$$

substituting Eq. A6 into Eq. A11,

$$\hat{B} = \frac{YD_R(R_{\text{in}} - \hat{R}(z_l))}{m(1-\varepsilon)(z_b + 1/h - z_l)} \tag{A12}$$

and finally setting $R(z_l) = R^*$ gives us the equilibrium biomass

$$\hat{B} = \frac{YD_R(R_{\text{in}} - R^*)}{m(1-\varepsilon)(z_b + 1/h - z_l)} \tag{A13}$$

The light level at and immediately under the layer is then given by Eq. A8,

$$\hat{I}(z_l) = I_{\text{in}} e^{-\frac{aYD_R(R_{\text{in}} - R^*)}{m(1-\varepsilon)(z_b+1/h-z_l)} - a_{\text{bg}} z_l} > I^* \tag{A14}$$

Note that $\hat{I}(z_l)$ decreases as z_l increases, so that a deeper nutrient-limited layer produces more shade.

In the case of light-limitation, evaluating Eq. A8 at z_l and rearranging gives us the equilibrium biomass,

$$\hat{B} = \frac{\log(I_{\text{in}}/I^*)}{a} - \frac{a_{\text{bg}}}{a} z_l \tag{A15}$$

The nutrient level at and above the layer is given by substituting Eq. A15 into Eq. A12, and rearranging:

$$\hat{R}(z_l) = R_{\text{in}} - \frac{m(1-\varepsilon)(z_b + 1/h - z_l)}{aYD_R} \left(\log(I_{\text{in}}/I^*) - a_{\text{bg}} z_l\right) > R^* \tag{A16}$$

Note that $\hat{R}(z_l)$ increases as z_l increases, so that a shallower light-limited layer depresses nutrient levels more.

In the case of co-limitation, both Eq. A13 and A15 hold, and $\hat{R}(z_l) = R^*$ and $\hat{I}(z_l) = I^*$.