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Appendix from C. A. Klausmeier and E. Litchman, "Successional Dynamics in the Seasonally Forced Diamond Food Web"
(Am. Nat., vol. 180, no. 1, p. 1)

Analytical Details

Case I

To determine which of these trajectories occur for a given ϕ, we first calculate the rate at which P_1 invades the [None] trajectory, averaged over a period

$$\Lambda_{P_1,[\mathrm{None}]} = \phi\lambda_{P_1,\varnothing,\mathrm{good}} + (1-\phi)\lambda_{P_1,\varnothing,\mathrm{bad}} \tag{A1}$$

(Klausmeier 2010). To find the critical ϕ above which P_1 can persist, we set $\Lambda_{P_1,[\mathrm{None}]} = 0$ and solve for ϕ_{crit,P_1}:

$$\phi_{\mathrm{crit},P_1} = \frac{\lambda_{P_1,\varnothing,\mathrm{bad}}}{\lambda_{P_1,\varnothing,\mathrm{bad}} - \lambda_{P_1,\varnothing,\mathrm{good}}}. \tag{A2}$$

To solve for the timing of the transition from $\varnothing$ to $\{P_1\}$ in the [P_1 only] trajectory, we note that P_1's decline in the bad season must equal its increase in the beginning of the good season:

$$0 = \lambda_{P_1,\varnothing,\mathrm{good}}\Delta_{\varnothing,\mathrm{good}} + \lambda_{P_1,\varnothing,\mathrm{bad}}\Delta_{\varnothing,\mathrm{bad}}, \tag{A3}$$

where $\Delta_{\varnothing,\mathrm{good}} = t_{\varnothing\to\{P_1\}}$ and $\Delta_{\varnothing,\mathrm{bad}} = 1-\phi$, which can be solved to find

$$t_{\varnothing\to\{P_1\}} = \frac{-(1-\phi)\lambda_{P_1,\varnothing,\mathrm{bad}}}{\lambda_{P_1,\varnothing,\mathrm{good}}}. \tag{A4}$$

Case II

To determine when the [P_1 & Z] trajectory occurs, we see when Z can invade the [P_1 only] trajectory. The invasion rate of Z into the [P_1 only] trajectory is the average of its growth rates in each state, weighted by the time spent in those states:

$$\Lambda_{Z,[P_1\,\mathrm{only}]} = \lambda_{Z,\varnothing,\mathrm{good}}\Delta_{\varnothing,\mathrm{good}} + \lambda_{Z,\{P_1\},\mathrm{good}}\Delta_{\{P_1\},\mathrm{good}} + \lambda_{Z,\varnothing,\mathrm{bad}}\Delta_{\varnothing,\mathrm{bad}}, \tag{A5}$$

where $\Delta_{\varnothing,\mathrm{good}} = t_{\varnothing\to\{P_1\}}$, $\Delta_{\{P_1\},\mathrm{good}} = \phi - t_{\varnothing\to\{P_1\}}$, $\Delta_{\varnothing,\mathrm{bad}} = 1-\phi$, and $t_{\varnothing\to\{P\}}$ is given by equation (A4). The critical ϕ for the Z's invasion into a monoculture of P_1 is found by setting $\Lambda_{Z,[P_1\,\mathrm{only}]} = 0$ and solving for $\phi_{\mathrm{crit},Z}$:

$$\phi_{\mathrm{crit},Z} = \frac{\lambda_{Z,\varnothing,\mathrm{bad}}\lambda_{P_1,\varnothing,\mathrm{good}} - \lambda_{Z,\varnothing,\mathrm{good}}\lambda_{P_1,\varnothing,\mathrm{bad}} + \lambda_{Z,\{P_1\},\mathrm{good}}\lambda_{P_1,\varnothing,\mathrm{bad}}}{\lambda_{Z,\varnothing,\mathrm{bad}}\lambda_{P_1,\varnothing,\mathrm{good}} - \lambda_{Z,\{P_1\},\mathrm{good}}\lambda_{P_1,\varnothing,\mathrm{good}} - \lambda_{Z,\varnothing,\mathrm{good}}\lambda_{P_1,\varnothing,\mathrm{bad}} + \lambda_{Z,\{P_1\},\mathrm{good}}\lambda_{P_1,\varnothing,\mathrm{bad}}}. \tag{A6}$$

In the [P_1 & Z] trajectory, there are two unknowns, $t_{\varnothing\to\{P_1\}}$ and $t_{\{P_1\}\to\{P_1,Z\}}$, and two equations, one from each species. They are

$$\begin{gathered}\lambda_{P_1,\varnothing,\mathrm{bad}}\Delta_{\varnothing,\mathrm{bad}} + \lambda_{P_1,\varnothing,\mathrm{good}}\Delta_{\varnothing,\mathrm{good}} = 0,\\ \lambda_{Z,\varnothing,\mathrm{bad}}\Delta_{\varnothing,\mathrm{bad}} + \lambda_{Z,\varnothing,\mathrm{good}}\Delta_{\varnothing,\mathrm{good}} + \lambda_{Z,\{P_1\},\mathrm{good}}\Delta_{\{P_1\},\mathrm{good}} = 0,\end{gathered} \tag{A7}$$

where $\Delta_{\{P_1\},\mathrm{good}} = t_{\{P_1\}\to\{P_1,Z\}} - t_{\varnothing\to\{P_1\}}$, which can be solved to find

$$\begin{aligned} t_{\varnothing\to\{P_1\}} &= \frac{-(1-\phi)\lambda_{P_1,\varnothing,\mathrm{bad}}}{\lambda_{P_1,\varnothing,\mathrm{good}}},\\ t_{\{P_1\}\to\{P_1,Z\}} &= \frac{-(1-\phi)(\lambda_{P_1,\varnothing,\mathrm{good}}\lambda_{Z,\varnothing,\mathrm{bad}} - \lambda_{P_1,\varnothing,\mathrm{bad}}\lambda_{Z,\varnothing,\mathrm{good}} + \lambda_{P_1,\varnothing,\mathrm{bad}}\lambda_{Z,\{P_1\},\mathrm{good}})}{\lambda_{P_1,\varnothing,\mathrm{good}}\lambda_{Z,\{P_1\},\mathrm{good}}}.\end{aligned} \tag{A8}$$

Case III

The critical ϕ that marks the transition between the [P_1 & Z] and [P_1, P_2, & Z] trajectories can be found by setting the invasion rate of P_2 into [P_1 & Z] equal to 0 and solving for ϕ_{crit,P_2}:

$$0 = \Lambda_{P_2,[P_1\&Z]} = \lambda_{P_2,\varnothing,\mathrm{good}}\Delta_{\varnothing,\mathrm{good}} + \lambda_{P_2,\{P_1\},\mathrm{good}}\Delta_{\{P_1\},\mathrm{good}} + \lambda_{P_2,\{P_1,Z\},\mathrm{good}}\Delta_{\{P_1,Z\},\mathrm{good}} + \lambda_{P_2,\varnothing,\mathrm{bad}}\Delta_{\varnothing,\mathrm{bad}}, \tag{A9}$$

where $t_{\varnothing\to\{P_1\}}$ and $t_{\{P_1\}\to\{P_1,Z\}}$ are given by equation (A8). The resulting algebraic expression (and others below) are too lengthy to publish and can be found in the Mathematica code. The timing of transitions in the [P_1, P_2, & Z] trajectory can be found by solving

$$\begin{aligned} 0 &= \lambda_{P_1,\varnothing,\mathrm{bad}}\Delta_{\varnothing,\mathrm{bad}} + \lambda_{P_1,\varnothing,\mathrm{good}}\Delta_{\varnothing,\mathrm{good}}, \\ 0 &= \lambda_{P_2,\varnothing,\mathrm{bad}}\Delta_{\varnothing,\mathrm{bad}} + \lambda_{P_2,\varnothing,\mathrm{good}}\Delta_{\varnothing,\mathrm{good}} + \lambda_{P_2,\{P_1\},\mathrm{good}}\Delta_{\{P_1\},\mathrm{good}} + \lambda_{P_2,\{P_1,Z\},\mathrm{good}}\Delta_{\{P_1,Z\},\mathrm{good}}, \\ 0 &= \lambda_{Z,\varnothing,\mathrm{bad}}\Delta_{\varnothing,\mathrm{bad}} + \lambda_{Z,\varnothing,\mathrm{good}}\Delta_{\varnothing,\mathrm{good}} + \lambda_{Z,\{P_1\},\mathrm{good}}\Delta_{\{P_1\},\mathrm{good}} \end{aligned} \tag{A10}$$

(see Mathematica code).

Case IV

Analysis of [P_2 & Z] is identical to that of [P_1 & Z] (“Case I”), except with different subscripts. Analytical expressions for the timings and critical ϕ’s of the [PEG] and [2 × P_2] trajectories can be easily found with a computer algebra program but are too lengthy to publish (see Mathematica code).

The stability of these trajectories is determined by the annual map that projects the communities ahead one period (Klausmeier 2010). In both cases, the annual map has three eigenvalues, two of which are 0 and the third given by $\Lambda = \partial P_1(t+1)/\partial P_1(t)$ (see Mathematica code). These nonzero eigenvalues can be solved analytically; they are

$$\Lambda_{[\mathrm{PEG}]} = \frac{\lambda_{P_1,\{P_2,Z\},\mathrm{good}}[\lambda_{Z,\varnothing,\mathrm{good}}(\lambda_{P_2,\{P_1\},\mathrm{good}} - \lambda_{P_2,\{P_1,Z\},\mathrm{good}}) + \lambda_{Z,\{P_1\},\mathrm{good}}(\lambda_{P_2,\{P_1,Z\},\mathrm{good}} - \lambda_{P_2,\varnothing,\mathrm{good}})]}{\lambda_{P_1,\varnothing,\mathrm{good}}\lambda_{P_2,\{P_1,Z\},\mathrm{good}}\lambda_{Z,\{P_1\},\mathrm{good}}}, \tag{A11}$$

$$\Lambda_{[2\times P_2]} = \frac{\lambda_{P_1,\{P_2,Z\},\mathrm{good}}[\lambda_{P_2,\{P_1,Z\},\mathrm{good}}(\lambda_{Z,\{P_1\},\mathrm{good}} - \lambda_{Z,\{P_2\},\mathrm{good}}) + \lambda_{P_2,\{P_1\},\mathrm{good}}\lambda_{Z,\{P_2\},\mathrm{good}}]}{\lambda_{P_1,\{P_2\},\mathrm{good}}\lambda_{P_2,\{P_1,Z\},\mathrm{good}}\lambda_{Z,\{P_1\},\mathrm{good}}}. \tag{A12}$$

Note that these Λ’s do not depend on the length of the growing season ϕ.

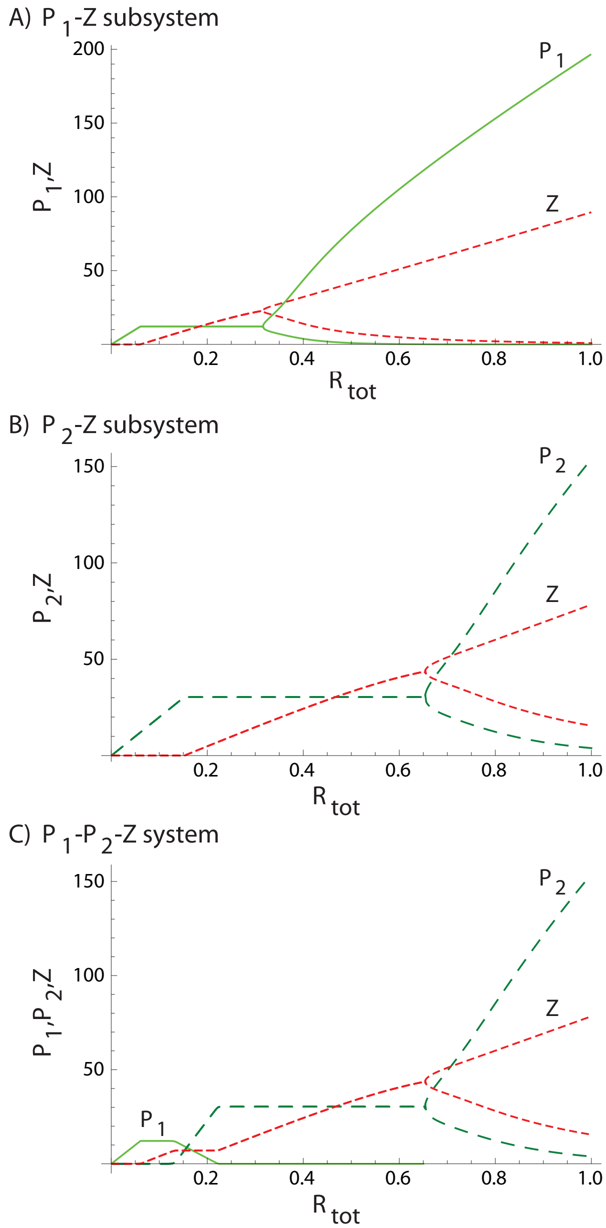

Figure A1: Bifurcation diagrams of the unforced model (eq. [1]): P_1 in light green, P_2 in dark green, and Z in red. *A*, P_1-Z subsystem. *B*, P_2-Z subsystem. *C*, Full P_1-P_2-Z system.

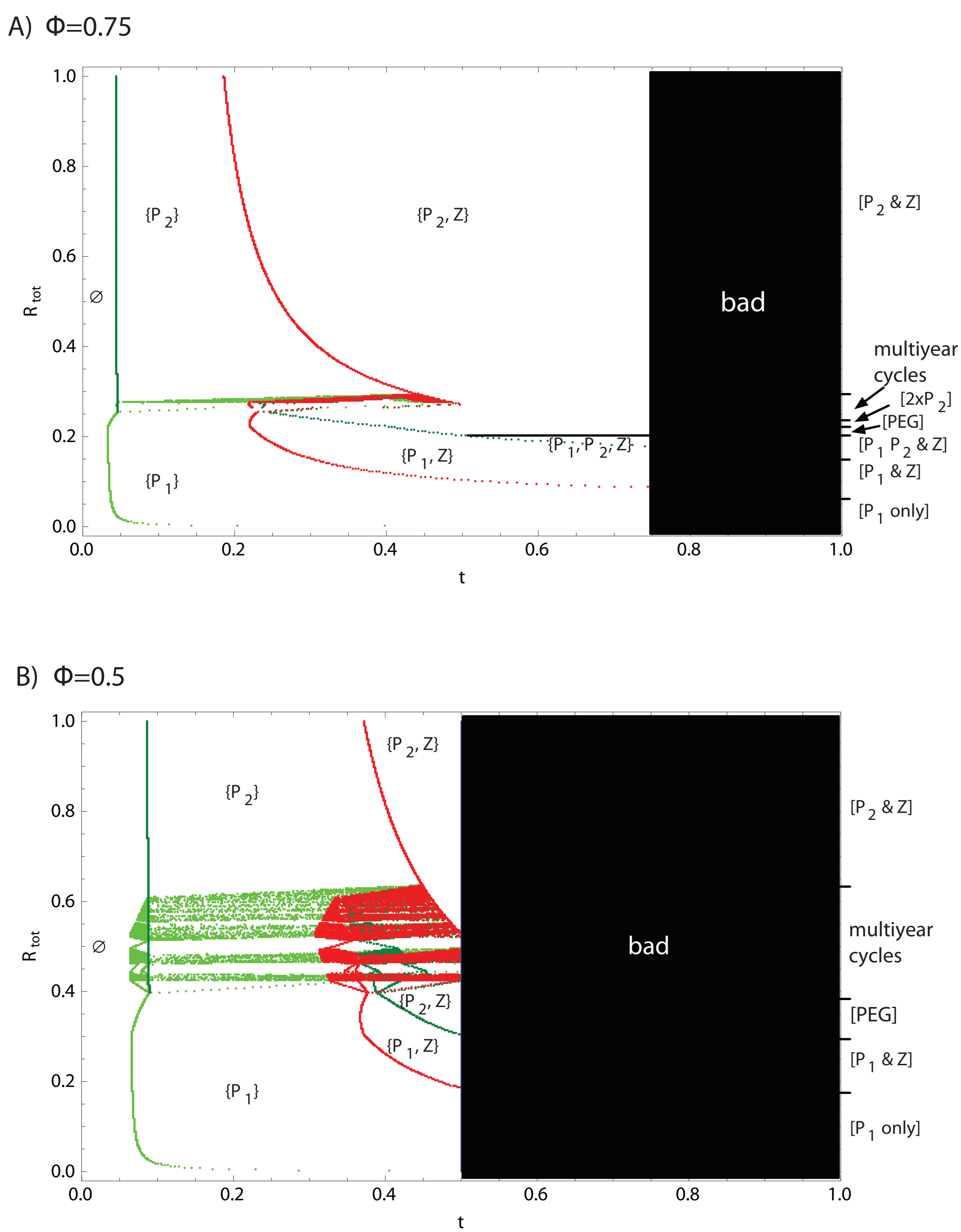

Figure A2: Effect of R_{tot} on timing of transitions for two ϕ values: 0.75 (*A*) and 0.5 (*B*).

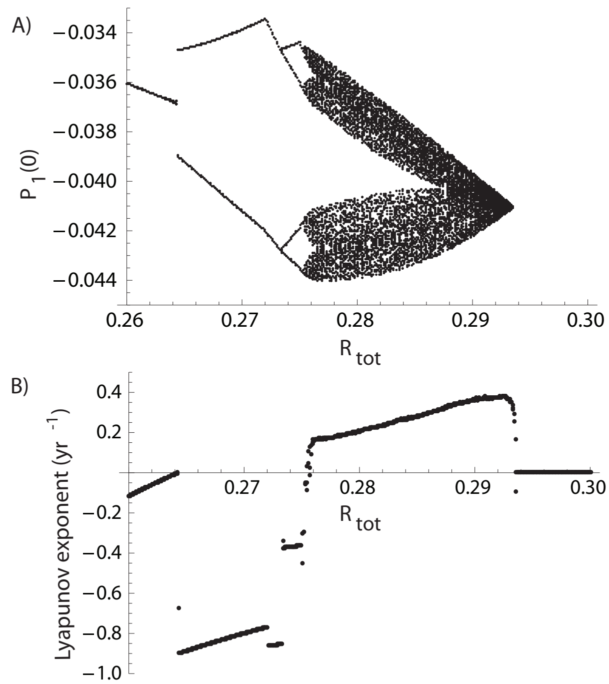

Figure A3: Detail of multiyear cycles. *A*, $P_1(0)$ versus R_{tot}. Note the period-doubling route to chaos as R_{tot} increases but "instant chaos" as R_{tot} decreases. *B*, Dominant Lyapunov exponent versus R_{tot}, calculated with the method of Sandri (1996).